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INHOMOGENEOUS DEFORMATIONS OF NON-LINEARLY ELASTIC WEDGES

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Abstract—We study a class of non-universal inhomogeneous deformations of both compressible and incompressible isotropic non-linearly elastic wedges. In the case of compressible solids we obtain a necessary condition which the stored energy ought to satisfy, so that it may be possible to deform the body in the assumed manner, by the application of surface tractions. In the case of the incompressible solids, we show that the pressure field ought to have a special structure if the assumed form of deformation is to be possible. For certain simple constitutive theories the governing equations can be integrated and explicit exact solutions can be obtained.

1. INTRODUCTION

Recently, there has been a great deal of interest in determining inhomogeneous solutions to boundary value problems in non-linear elasticity. Ericksen (1955) proved that all universal deformations† in compressible elastic materials, in the absence of body forces, are homogeneous. Earlier he had shown that five classes of inhomogeneous universal deformations are possible in the case of incompressible isotropic elastic materials (Ericksen, 1954). This result was subsequently extended to more general simple materials by Carroll (1967), Wineman (1967) and Fosdick (1968). Universal deformations and motions play a special role in both fluid and solid mechanics in that they suggest experiments in which the deformation of the body under consideration is known from the outset. Also, many of the simple and elegant exact solutions in both solid and fluid theories are universal. However, there is no reason to expect all materials, say for instance rubber and steel, to respond in an identical manner. Hence, it would seem more reasonable for one to ask the following question: given a non-homogeneous deformation, what class of constitutive equations can support such a deformation?

In the light of the above remarks, it is not surprising that a considerable amount of interest has been evinced recently in the study of inhomogeneous deformations of isotropic non-linear elastic materials (cf. Sensenig, 1965; Holden, 1968; Varley and Cumberbatch, 1977; Ogden, 1977; Isherwood and Ogden, 1977; Currie and Hayes, 1981; Rajagopal and Wineman, 1985; Rajagopal *et al.*, 1986; Chao *et al.*, 1987; McLeod *et al.*, 1988; Abeyaratne and Horgan, 1984; Horgan, 1989; Chung *et al.*, 1986; Carroll, 1988).

In this paper we study a class of non-universal inhomogeneous deformations which is possible in subclasses of compressible and incompressible isotropic elastic materials. That is, the deformations share a common structure when expressed in terms of certain arbitrary functions, but these differ from material to material. The tractions which are necessary to effect and support the deformation under consideration can be computed, and these will also differ from material to material.

We shall consider the inhomogeneous squeezing or fanning or non-linear elastic wedges. For the assumed form of the deformation, the equilibrium equations reduce to a system of coupled non-linear ordinary differential equations, and thus given a specific

† A deformation is universal, for a given class of bodies subject to a given body force **b**, if it satisfies the equations of motion (or equilibrium) for all the bodies belonging to the class.

constitutive model we can discuss the existence of a solution (or solutions) to the problem under consideration. In the case of isotropic compressible hyperelastic materials we obtain a necessary condition which the stored energy function ought to satisfy in order that there exists a solution to the boundary value problem under consideration. For simple constitutive theories, these equations can be integrated and explicit exact solutions established (cf. Tao and Rajagopal, 1991; Rajagopal and Tao, in press; Tao and Rajagopal, 1990).

In Section 2 the inhomogeneous deformation is introduced and the basic kinematics are discussed. The following section is devoted to a discussion on isotropic elastic materials. The deformation of incompressible isotropic materials is discussed in Section 4 and the final section deals with a study of compressible elastic materials within the context of the inhomogeneous deformation under consideration.

2. KINEMATICS

We consider the class of deformations

$$r = A(\Theta)R,\quad \theta = B(\Theta),\quad z = Z, \tag{1}$$

where (R, Θ, Z) and (r, θ, z) denote the reference and current coordinates of the same particle, in a cylindrical coordinate system.

The deformation under consideration is appropriate for the inhomogeneous squeezing of a wedge wherein each radial ray emanating from the apex of the wedge remains radial, the stretch being $A(\Theta)$. The deformation gradient $\mathbf{F}$ has the following matrix representation for its physical components:

$$\mathbf{F} = \begin{pmatrix} A & A' & 0 \\ 0 & AB' & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where the prime denotes differentiation with respect to Θ. Thus, referred to an orthonormal basis the deformation gradient $\mathbf{F}$ has the representation

$$\mathbf{F} = A\mathbf{e}_r \otimes \mathbf{e}_R + A'\mathbf{e}_r \otimes \mathbf{e}_\Theta + AB'\mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z. \tag{3}$$

It follows from (1) that

$$R\frac{\partial}{\partial R} = r\frac{\partial}{\partial r},\quad \frac{\partial}{\partial \Theta} = \frac{A'}{A}r\frac{\partial}{\partial r} + B'\frac{\partial}{\partial \theta}, \tag{4}$$

and hence

$$r\frac{\partial}{\partial r} = R\frac{\partial}{\partial R},\quad \frac{\partial}{\partial \theta} = \frac{-A'}{AB'}R\frac{\partial}{\partial R} + \frac{1}{B'}\frac{\partial}{\partial \Theta}. \tag{5}$$

A simple calculation then gives

$$R\,\mathrm{Grad}\,\mathbf{F} = (F\mathbf{e}_r + G\mathbf{e}_\theta) \otimes \mathbf{e}_\Theta \otimes \mathbf{e}_\Theta. \tag{6}$$

where

$$F(\Theta) := A'' + A - A(B')^2, \tag{7}$$

$$G(\Theta) := AB'' + 2A'B'. \tag{8}$$

Thus, the condition for homogeneous deformation is

$$F(\Theta) = A'' + A - A(B')^2 = 0, \tag{9}$$

$$G(\Theta) = AB'' + 2A'B' = 0. \tag{10}$$

The Cauchy–Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{B}^{-1}$ have the physical components

$$\mathbf{B} = \begin{pmatrix} A^2 + (A')^2 & AA'B' & 0 \\ AA'B' & A^2(B')^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

and

$$\mathbf{B}^{-1} = \frac{1}{A^4(B')^2}\begin{pmatrix} A^2(B')^2 & -AA'B' & 0 \\ -AA'B' & A^2 + (A')^2 & 0 \\ 0 & 0 & A^4B'^2 \end{pmatrix}, \tag{12}$$

respectively.

The principal invariants of $\mathbf{B}$ are

$$I_1 = \operatorname{tr} \mathbf{B} = A^2 + (A')^2 + A^2(B')^2 + 1, \tag{13}$$

$$I_2 = I_1 + I_3 - 1 = A^2 + A'^2 + A^2(B')^2 + A^4(B')^2, \tag{14}$$

$$I_3 = A^4(B')^2. \tag{15}$$

Integration of (10) gives that I_3 = constant, and substitution for B' from (15) into (9), and integrating, gives that I_1 is a constant. Thus, as expected, homogeneous deformation implies constant invariants. On the other hand, if the principal invariants are constant we have

$$I'_1 = 2A'(A + A'' - A(B')^2) + 2AB'(AB'' + 2A'B') = 0 \tag{16}$$

and

$$I'_3 = 2A^3B'(AB'' + 2A'B') = 0. \tag{17}$$

Since $AB' \neq 0$, (17) implies (10), and (16) then implies that either (9) or $A' = 0$. Thus, constant invariants implies that we have either homogeneous deformation, or A = constant and B' = constant.

The deformation

$$r = \alpha R, \quad \theta = \beta\Theta, \quad z = Z \tag{18}$$

has constant invariants and is not homogeneous if $\beta \neq 1$ (cf. Fosdick, 1968).

For isochoric deformation, (2) implies that

$$A^2B' = 1. \tag{19}$$

Substituting from (19) in (9) and (10) shows that for homogeneous deformation we need

$$F = A'' + A - \frac{1}{A^3} = 0, \tag{20}$$

and the principal invariants become

$$I_1 = I_2 = A^2 + (A')^2 + \frac{1}{A^2} + 1 = I, \quad \text{and} \quad I_3 = 1. \tag{21}$$

3. ISOTROPIC ELASTIC MATERIALS

In the case of an elastic material the strain energy function W is a function of the deformation gradient $\mathbf{F}$, i.e.

$$W = W(\mathbf{F}). \tag{22}$$

The Piola stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}. \tag{23}$$

If the solid is to be isotropic, then restrictions due to frame indifference and material symmetry imply that the strain energy function depends on F only through the principal invariants:

$$W = \hat{W}(I_1, I_2, I_3). \tag{24}$$

The equations of equilibrium, in the absence of body forces, reduces to

$$\text{Div}\,\mathbf{S} = 0, \tag{25}$$

where the operator Div denotes the divergence operator with respect to the reference coordinates.

It follows from (23) and (25) that

$$\frac{\partial^2 W}{\partial \mathbf{F}^2} \cdot \text{Grad}\,\mathbf{F} = 0, \tag{26}$$

or, in rectangular Cartesian component form

$$\frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{j,B}} x_{j,AB} = 0. \tag{27}$$

In the case of incompressible solids, the Piola–Kirchhoff stress tensor is given by

$$\mathbf{S} = -p(\mathbf{F}^{-1})^T + \frac{\partial W}{\partial \mathbf{F}}, \tag{28}$$

where p is arbitrary, and

$$W = \hat{W}(I_1, I_2) \tag{29}$$

since

$$I_3 = 1. \tag{30}$$

If follows from (30) that $\operatorname{Div}(\mathbf{F}^{-1})^T = 0$, and substitution from (28) into (25) gives

$$\operatorname{grad} p = \operatorname{Div}\frac{\partial W}{\partial \mathbf{F}} = \frac{\partial^2 W}{\partial \mathbf{F}^2}\cdot \operatorname{Grad}\mathbf{F}, \tag{31}$$

or in indicial form

$$p_{,i} = \frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{j,B}} x_{j,AB}. \tag{32}$$

4. INCOMPRESSIBLE ELASTIC SOLIDS

An incompressible material can undergo only isochoric deformations, and hence

$$A^2B' = 1. \tag{33}$$

Substitution of (33) into (2) and (11) gives

$$\mathbf{F} = \begin{bmatrix} A & A' & 0 \\ 0 & \dfrac{1}{A} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{34}$$

and

$$\mathbf{B} = \begin{bmatrix} A^2 + A'^2 & \dfrac{A'}{A} & 0 \\ \dfrac{A'}{A} & \dfrac{1}{A^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

It follows that the equations of equilibrium (30) and (31) become

$$\frac{\partial p}{\partial r} = \frac{\partial^2 W}{\partial F_{r\Theta}^2}\frac{F}{R}, \tag{36}$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{\partial^2 W}{\partial F_{r\Theta}\,\partial F_{\theta\Theta}}\frac{F}{R}. \tag{37}$$

On making use of (4), (5) and (7), we obtain

$$R\frac{\partial p}{\partial R} = \frac{\partial^2 W}{\partial F_{r\Theta}^2} A\left(A'' + A - \frac{1}{A^3}\right), \tag{38}$$

$$\frac{\partial p}{\partial \Theta} = \frac{\partial^2 W}{\partial F_{r\Theta}\,\partial F_{\theta\Theta}}\frac{1}{A}\left(A'' + A - \frac{1}{A^3}\right) + \frac{A'}{A} R\frac{\partial p}{\partial R}. \tag{39}$$

By virtue of (38) and (39), $\partial p/\partial \Theta$ is independent of R, and (38) then gives

$$R\frac{\partial p}{\partial R} = \frac{\partial^2 W}{\partial F_{r\Theta}^2} A\left(A'' + A - \frac{1}{A^3}\right) = K = \text{constant}. \tag{40}$$

Hence,

$$\frac{\partial p}{\partial \Theta} = \frac{\partial^2 W}{\partial F_{r\Theta}\,\partial F_{\theta\Theta}} \frac{1}{A}\left(A'' + A - \frac{1}{A^3}\right) + K\frac{A'}{A}. \tag{41}$$

It follows from (40) that the pressure field p has the form

$$p = K \ln R + \Psi(\Theta), \tag{42}$$

where Ψ is some arbitrary function of Θ.

We now make use of (29), and since $I_1 = I_2 = I$ say, in plane strain, we write

$$\omega(I) = \hat{W}(I, I). \tag{43}$$

We also have

$$I = F_{rR}^2 + F_{r\Theta}^2 + F_{\theta R}^2 + F_{\theta\Theta}^2 + 1, \tag{44}$$

so that

$$\frac{\partial^2 W}{\partial F_{r\Theta}^2} = 2\frac{\mathrm{d}\omega}{\mathrm{d}I} + 4\frac{d^2\omega}{dI^2}F_{r\Theta}^2, \tag{45}$$

$$\frac{\partial^2 W}{\partial F_{r\Theta}\,\partial F_{\theta\Theta}} = 4\frac{\mathrm{d}^2\omega}{\mathrm{d}I^2}F_{r\Theta}F_{\theta\Theta}. \tag{46}$$

Using (44)–(46) and (34) in (40) and (41) yields

$$R\frac{\partial p}{\partial R} = 2\left\{\frac{\mathrm{d}\omega}{\mathrm{d}I} + 2\frac{\mathrm{d}^2\omega}{\mathrm{d}I^2}(A')^2\right\}A\left(A'' + A - \frac{1}{A^3}\right) = K, \tag{47}$$

$$\frac{\partial p}{\partial \Theta} = 4\frac{\mathrm{d}^2\omega}{\mathrm{d}I^2}\frac{A'}{A^2}\left(A'' + A - \frac{1}{A^3}\right) + K\frac{A'}{A}. \tag{48}$$

When $K = 0$ we recover the homogeneous deformation $F = 0$. Non-zero values of K give rise to inhomogeneous deformations with A satisfying (47) and (21).

We shall next discuss a couple of specific examples. Let us consider the deformation of a Mooney–Rivlin material. For a Mooney–Rivlin material undergoing plane strain deformation the strain energy function

$$\omega = \frac{\mu}{2}(I - 3), \tag{49}$$

where μ is the shear modulus for infinitesimal deformation. Thus

$$\frac{\partial^2\omega}{\partial F_{r\Theta}^2} = \mu, \quad \frac{\partial^2\omega}{\partial F_{r\Theta}\,\partial F_{\theta\Theta}} = 0, \tag{50}$$

and (47) and (48) reduce to

$$R\frac{\partial p}{\partial R} = \mu A\left(A''+A-\frac{1}{A^3}\right) = K, \tag{51}$$

$$\frac{\partial p}{\partial \Theta} = K\frac{A'}{A}, \tag{52}$$

with appropriate boundary conditions.

Fu *et al.* (1990) have investigated the consequences of eqns (51) and (52) subject to the boundary conditions:

$$A(\alpha) = A_0, \quad B(\alpha) = \beta_0, \quad B(-\alpha) = \beta_1. \tag{53}$$

When $K = 0$, they were able to integrate the equations to obtain

$$A^2(\Theta) = \left[C_1+k\sin\left\{2\arctan\left[\frac{\tan(\Theta+C_2)-k}{C_1}\right]\right\}\right]^{-1}. \tag{54}$$

$$B(\Theta) = \arctan\left[\frac{\tan(\Theta+C_2)-k}{C_1}\right]+C_3. \tag{55}$$

While the above solution seems at first sight to be quite involved, it is easy to show that (54) and (55) imply the deformation (1) takes the form

$$x = \lambda_1 X+\lambda_2 Y, \quad y = \lambda_3 X+\lambda_4 Y, \quad z = Z,$$

i.e. the deformation is homogeneous. When $K \neq 0$, Fu *et al.* (1990) find inhomogeneous solutions. Due to the nature of the equations they are able to carry out a phase plane analysis of the solution. An interesting feature of the problem is that for certain wedge angles, and appropriate surface tractions, the deformation of the wedge is not unidirectional, that is, there are subdomains of the wedge wherein material elements move towards the apex in some of them and away from the apex in others.

Next, let us consider the case of a generalized power-law Neo-Hookean solid. The strain energy function for such a material has the form

$$W = \frac{\mu}{2}\left\{\left[1+\frac{b}{n}(I_1-3)\right]^n-1\right\} \tag{56}$$

and thus the Cauchy–Stress $\mathbf{T}$ is given by

$$\mathbf{T} = -p\mathbf{1}+\mu\left[1+\frac{b}{n}(I_1-3)\right]^{n-1}\mathbf{B}, \tag{57}$$

where $\mu > 0$ is the shear modulus and b and n are positive numbers. When $n = 1$ the model reduces to that of a Neo-Hookean solid. It is well known that the equations of equilibrium for a power-law material modeled by (56) can lose ellipticity in plane strain when the power-law exponent $n < 1/2$. In the case of such a material, the pressure field takes the form

$$p(R,\Theta) = \mu a\ln R+\mu[1+(\lambda)^{-1}(I_1-3)]^{n-1}\frac{1}{A^2}+2\mu\int_{-\alpha}^{\Theta}[1+(\lambda)^{-1}(I_1-3)]^{n-1}\frac{A'}{A^3}\,d\bar{\Theta}+C_1. \tag{58}$$

where $\lambda = \mathbf{n} \cdot \mathbf{b}$, and a and C_1 are constants, and 2α is the angle of the undeformed wedge. Notice that (58) has the same form as predicted by (42).

Rajagopal and Tao (in press) have studied the deformations in such a wedge. Even when $a = 0$, which corresponds to a bounded pressure field, they find inhomogeneous solutions which have a "boundary layer structure" in the sense that adjacent to the boundary the deformation is inhomogeneous, while in the core region the deformation is homogeneous. They also found inhomogeneous solutions which correspond to the pressure field which varies logarithmically with the radial coordinate.

5. COMPRESSIBLE ELASTIC SOLIDS

We now examine the possibility of inhomogeneous deformations of the form (1), i.e. plane deformations for which radial lines remain radial lines and undergo uniform stretch, in compressible isotropic elastic solids. In the incompressible case, such inhomogeneous solutions are generally associated with a radial term in the pressure. However, as we discussed earlier in the power-law Neo-Hookean solid, inhomogeneous solutions are possible even when there is no radial dependence in the pressure. The stresses corresponding to a deformation of the type (1) in a compressible solid cannot exhibit a radial dependence since the physical components of $\mathbf{F}$ are a function of Θ only, and thus we might not expect to find inhomogeneous solutions. However, as we shall see, this is not so.

By virtue of (6), the equations of equilibrium (27) take the form

$$\frac{\partial^2 W}{\partial F_{r\Theta}^2} F + \frac{\partial^2 W}{\partial F_{r\Theta}\, \partial F_{\theta\Theta}} G = 0, \tag{59}$$

$$\frac{\partial^2 W}{\partial F_{r\Theta}\, \partial F_{\theta\Theta}} F + \frac{\partial^2 W}{\partial F_{\theta\Theta}^2} G = 0. \tag{60}$$

Although this is not a linear system, it is clear that one solution is the homogeneous solution $F = 0$ and $G = 0$ and that inhomogeneous solutions are possible only if

$$\frac{\partial^2 W}{\partial F_{r\Theta}^2} \frac{\partial^2 W}{\partial F_{\theta\Theta}^2} - \left(\frac{\partial^2 W}{\partial F_{r\Theta}\, \partial F_{\theta\Theta}}\right)^2 = 0. \tag{61}$$

If (59) and (60) have a non-trivial solution, then, there exist vectors $\mathbf{n}$ and $\mathbf{N}$ such that

$$\frac{\partial^2 W}{\partial x_{i,A}\, \partial x_{j,B}} n_i N_A n_j N_B = 0. \tag{62}$$

Indeed, from (59) and (60), we may take

$$\mathbf{n} = F\mathbf{e}_r + G\mathbf{e}_\theta; \quad \mathbf{N} = \mathbf{e}_\Theta. \tag{63}$$

Equation (62) implies that the acoustic tensor is not positive-definite.

Next, we shall investigate specific materials which meet the requirement (61) which allows for inhomogeneous solutions. Let us consider the Blatz–Ko material for which the strain energy function W has the form

$$W = \frac{\mu}{2}\left[\frac{I_2}{I_3} + 2(I_3)^{1/2} - 5\right], \tag{64}$$

where μ is the shear modulus. A lengthy but straightforward calculation shows that the equations of equilibrium reduce to (cf. Tao and Rajagopal, 1990)

$$-\frac{d}{d\Theta}\left[\frac{A'}{(B')^2A^5}\right]+\frac{1}{A^4}-\frac{1}{A^4(B')^2}-\frac{(A')^2}{(B')^2A^6}=0, \tag{65}$$

and

$$\frac{d}{d\Theta}\left[\frac{1}{A^4(B')^3}+\frac{(A')^2}{A^6(B')^3}\right]-2\frac{A'}{A^5(B')}=0. \tag{66}$$

Defining

$$p:=-4\frac{A'}{A},\quad q:=\frac{1}{A^4}\quad\text{and}\quad s:=\frac{1}{(B')^2}. \tag{67}$$

(65) and (66) can be re-written as

$$\tfrac{1}{4}q''s+\tfrac{1}{4}q's'+q-qs-\tfrac{1}{16}q^{-1}(q')^2s=0, \tag{68}$$

and

$$[s(\tfrac{1}{4}p'-1-16p^2)+1]\left(1-\frac{p^2}{48}\right)=0. \tag{69}$$

If the solution is to be inhomogeneous, then the necessary condition (61) will have to be met. A simple computation shows that (61) implies that $A(\Theta)$ has to satisfy the requirement

$$\frac{-A'}{A}=\pm\sqrt{3} \tag{70}$$

and thus

$$A(\Theta)=\alpha\,e^{\pm\sqrt{3}\Theta}. \tag{71}$$

However, $A(\Theta)$ should also satisfy the equations of equilibrium (68) and (69). By virtue of (67) and (71), it follows that

$$\left(1-\frac{p^2}{48}\right)=0, \tag{72}$$

and thus $A(\Theta)$ given by (71) satisfies the equation of equilibrium (69). Once $A(\Theta)$ has been determined, we can go ahead and find $B(\Theta)$ from (68). Notice that the solution (71) is not a symmetric solution about the ray $\Theta=0$, and thus the tractions which have to be applied on the boundaries of the wedge have to be different at $\Theta=+\alpha$ and $\Theta=-\alpha$.

Next, we shall consider the class of Hadamard materials. Such materials are defined by a strain energy function of the form

$$W=b_1(I_1-3)+b_2(I_2-3)+h(I_3), \tag{73}$$

where b_1 and b_2 are constants and h is an arbitrary function of I_3.

For the class of Hadamard materials a straightforward computation shows that the equations of equilibrium reduce to

$$\mu F = 0, \tag{74}$$

$$\{\mu + 2C_2A^2 + 2A^2h'(I_3) + 4A^6B'^2h''(I_3)\}G = 0, \tag{75}$$

where

$$\frac{\partial^2 W}{\partial F_{r\Theta}^2} = \mu. \tag{76}$$

Notice that $F = 0$ and $G = 0$, i.e. the homogeneous deformation, is the solution unless

$$\mu + 2C_2A^2 + 2A^2h'(I_3) + 4A^6B'^2h''(I_3) = 0. \tag{77}$$

Thus, inhomogeneous deformations are possible in materials wherein the function h satisfies the above equation subject to the appropriate boundary conditions.

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REFERENCES

Abeyaratne, R. and Horgan, C. O. (1984). The pressurized hollow sphere problem in finite elastostatics for a class of compressible materials. *Int. J. Solids Structures* **20**, 715–723.
Carroll, M. M. (1967). Controllable deformations of incompressible simple materials. *Int. J. Engng Sci.* **5**, 515–525.
Carroll, M. M. (1988). Finite strain solutions incompressible isotropic elasticity. *J. Elast.* **20**, 65–92.
Chao, R. M., Rajagopal, K. R. and Wineman, A. S. (1987). Nonhomogeneous tension torsion of neo-Hookean and Mooney Rivlin materials. *Arch. Mech.* **39**, 551–559.
Chung, D. T., Horgan, C. O. and Abeyaratne, R. (1986). The finite deformation of internally pressurized hollow cylinders and spheres for a class of compressible elastic materials. *Int. J. Solids Structures.* **22**, 1557–1570.
Currie, P. K. and Hayes, M. (1981). On non-universal finite elastic deformations. In *Proceedings of the IUTAM Symposium on Finite Elasticity* (Edited by D. E. Carlson and R. T. Shield), 143.50. Martinus Nijhoff, The Hague.
Ericksen, J. L. (1954). Deformations possible in every isotropic incompressible perfectly elastic body. *Z. Angew. Math. Phys.* **5**, 466–486.
Ericksen, J. L. (1955). Deformations possible in every compressible isotropic perfectly elastic material. *J. Math. Phys.* **34**, 126–128.
Fosdick, R. L. (1968). Dynamically possible motions of incompressible, isotropic, simple materials. *Arch. Ration. Mech. Anal.* **29**, 272–288.
Fu, D., Rajagopal, K. R. and Szeri, A. Z. (1990). Non-homogeneous deformations in a wedge of Mooney Rivlin material. *Int. J. Non-linear Mech.* **25**, 1–13.
Holden, J. T. (1968). A class of exact solutions for finite plane strain deformations of a particular elastic material. *Appl. Sci. Res.* **19**, 171–181.
Horgan, C. O. (1989). Some remarks on axisymmetric solutions in finite elastostatics for compressible materials. *Proc. R. Irish Acad.* **A89**, 185–193.
Isherwood, D. A. and Ogden, R. W. (1977). Towards the solution of finite plane strain problems for compressible elastic solids. *Int. J. Solids Structures* **13**, 105–123.
McLeod, J. B., Rajagopal, K. R. and Wineman, A. S. (1988). On the existence of a class of deformations for incompressible isotropic elastic materials. *Proc. R. Irish Acad.* **88A**, 91–101.
Ogden, R. W. (1977). Plane deformations of incompressible isotropic elastic solids: an integral equation formulation. *Mech. Res. Commun.* **4**, 347–352.
Rajagopal, K. R. and Tao, L. (in press). On an inhomogeneous deformation of a generalized Neo-Hookean material. *J. Elast.*
Rajagopal, K. R., Troy, W. and Wineman, A. S. (1986). A note on non-universal deformations of non-linearly elastic layers. *Proc. R. Irish Acad.* **86A**, 107–114.
Rajagopal, K. R. and Wineman, A. S. (1985). New exact solutions in non-linear elasticity. *Int. J. Engng Sci.* **23**, 217–234.
Sensenig, C. B. (1965). Non-linear theory for the deformation of prestressed circular plates and rings. *Commun. Pure Appl. Math.* **18**, 147–156.
Tao, L. and Rajagopal, K. R. (1990). On an inhomogeneous deformation of a Blatz-Ko material. **42**, 729–734.
Varley, E. and Cumberbatch, E. (1977). The finite deformation of an elastic material surrounding an elliptic hole. In *Finite Elasticity* (Edited by R. S. Rivlin), pp. 51–64. ASME, New York.
Wineman, A. S. (1967). *Universal deformations of incompressible simple materials.* University of Michigan Technical Report, Ann Arbor.